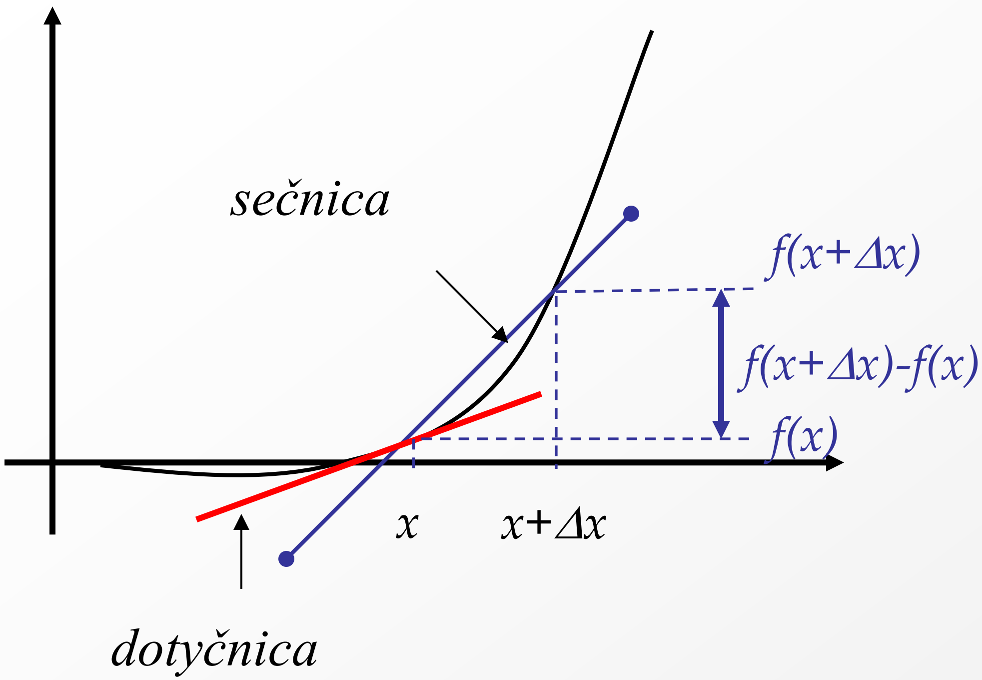
Vedecko-technické výpočty

Derivácia funkcií

Prehľad

- Definícia derivácie
- Numerické vyhodnotenie derivácie
- Presnosť výpočtu derivácie
- Derivácie vyšších rádov
- Aplikácia na diskrétne body

Definícia derivácie



$$f(x) = 5x^2$$

$$f'(x) = 5 \cdot 2x = 10x$$

$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f(x) = 5x^2$$

$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{5(x+\Delta x)^2 - 5x^2}{\Delta x}$$

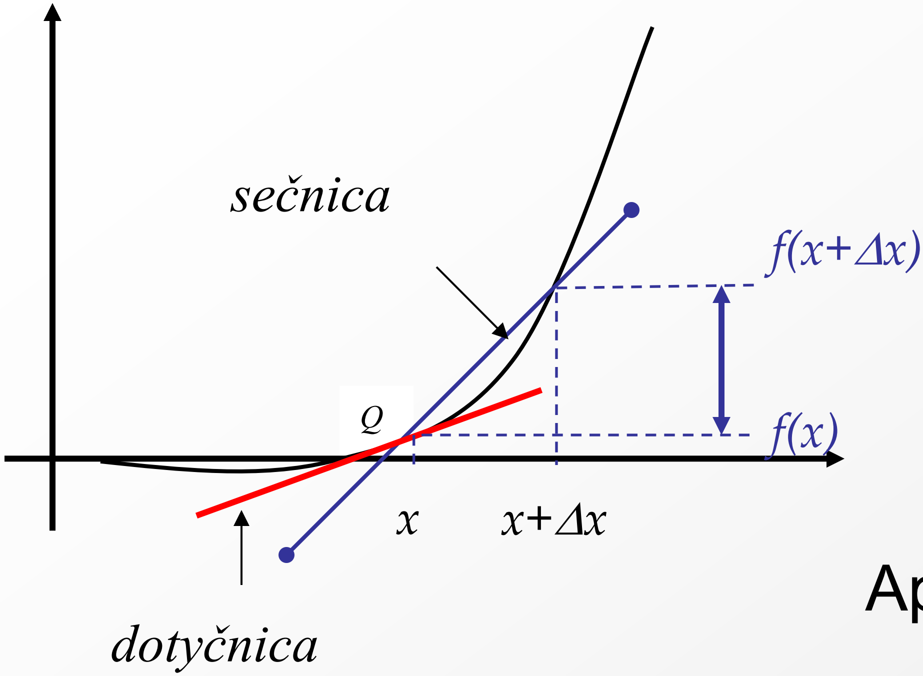
$$= \lim_{\Delta x \longrightarrow 0} \frac{5(x^2 + \Delta x^2 + 2x\Delta x) - 5x^2}{\Delta x}$$

$$= \lim_{\Delta x \longrightarrow 0} \frac{5\Delta x^2 + 10x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \longrightarrow 0} 5\Delta x + 10x$$

$$= 0 + 10x$$

Numerická derivácia (úvod)



$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Aproximujeme matematický prístup

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Konkrétny príklad

$$f(x) = 2e^{1.5x}$$

$$f'(3) = ???$$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Δx = 0.1

$$f'(3) \approx \frac{f(3+0.1) - f(3)}{0.1}$$

$$f'(3) \approx \frac{2e^{1.5 \cdot 3.1} - 2e^{1.5 \cdot 3}}{0.1}$$

$$f'(x) \approx 291.35$$

Odhad chýb podľa analytického vyjadrenie derivácie

$$\left| \frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \right| \Rightarrow f'(x) = 2(1.5e^{1.5x}) = 3e^{1.5 \cdot 3} = 270.05$$

$$E_t = 293.35 - 270.05 = 21.3$$

$$E_r = 7.8\%$$

Vplyv Δx

$$f(x) = 2e^{1.5x}$$

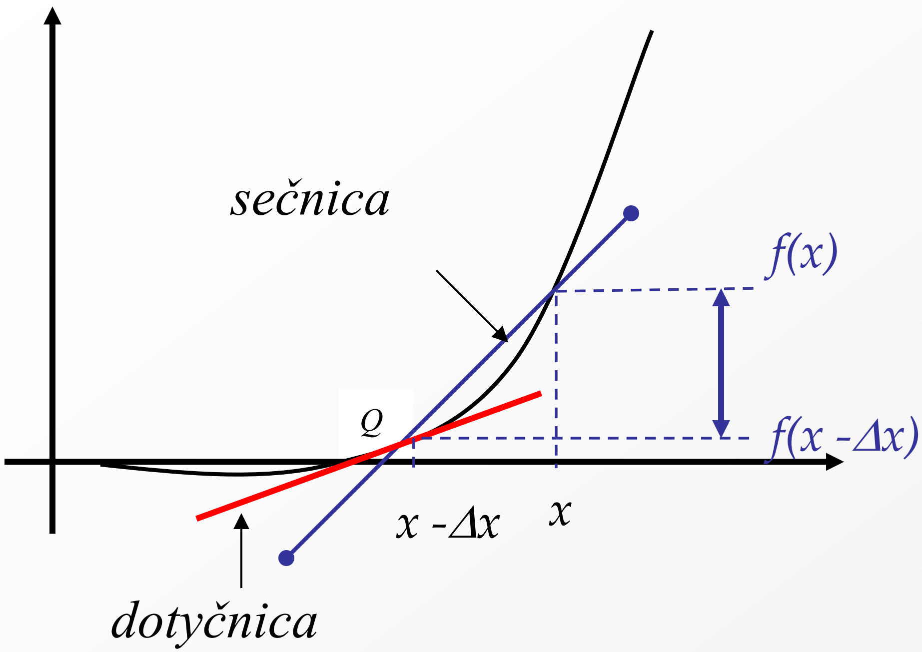
$$f'(3)_{real} = 270.05$$

Δx	$f'(x)$	E_t
0.1	291.35	21.30
0.05	280.43	10.38
0.025	275.18	5.127

Otvorené otázky:

1) Prečo polovičným intervalom zodpovedajú polovičné hodnoty chyby?
2) Ako vyhodnocovať vo všeobecnosti relatívnu chybu aby sme ohodnotili presnosť nášho výsledku?

Alternatívny prístup



Δx	$f'(x)$	E_t
0.1	250.77	-19.27
0.05	260.17	-9.88
0.025	265.05	-5.00

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$f(x) = 2e^{1.5x}$$

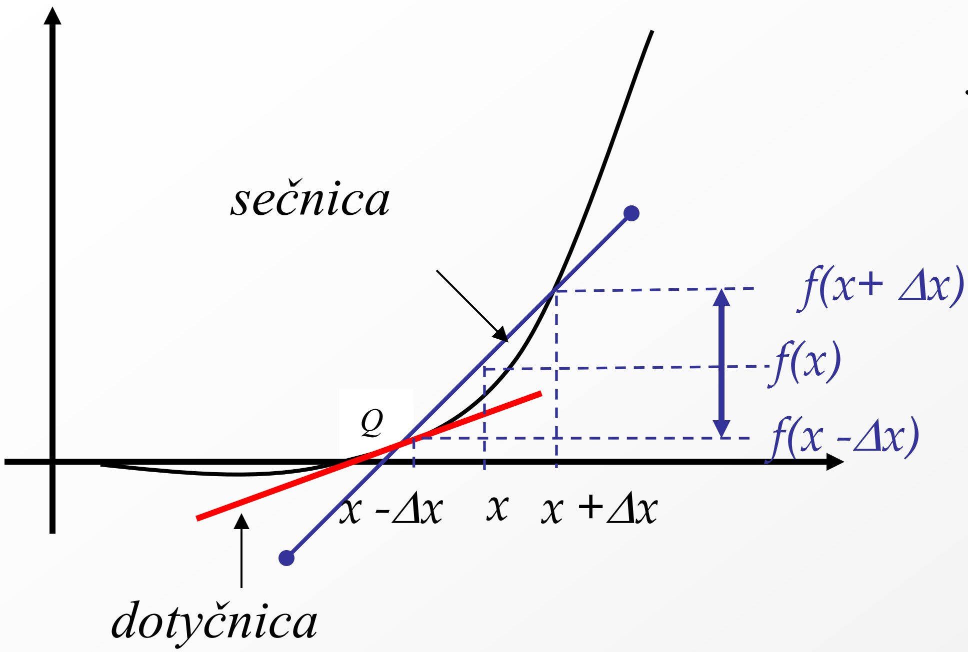
$$f'(3) = ???$$

$\Delta x = 0.1$

$$f'(3) \approx \frac{f(3) - f(3 - 0.1)}{0.1} = \frac{2e^{1.5\cdot 3} - 2e^{1.5\cdot 2.9}}{0.1} = 250.77$$

Stanislav.Antalic@fmph.uniba.sk

Centrálny prístup



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

$$f(x) = 2e^{1.5x}$$

$$f'(3) = ???$$

$\Delta x = 0.1$

$$f'(3) \approx \frac{f(3+0.1) - f(3-0.1)}{2 \cdot 0.1}$$

$$= \frac{2e^{1.5 \cdot 3.1} - 2e^{1.5 \cdot 2.9}}{0.2} = 271.06$$

$$E_t = 270.05 - 271.06 = 1.01$$

$$E_r = 0.375\%$$

Δx	$f'(x)$	E_t
0.1	271.06	-1.014
0.05	270.30	-0.2532
0.025	270.11	-0.0633

Polovičným intervalom zodpovedajú štvrtinové hodnoty chyby?

Presnosť výpočtu

Zoberme si Taylorov rozvoj

$$f(x+\Delta x)=f(x)+f'(x)\Delta x+\frac{f''(x)}{2!}(\Delta x)^2+\frac{f'''(x)}{3!}(\Delta x)^3+\frac{f''''(x)}{4!}(\Delta x)^4\ldots$$

$$f(x+\Delta x)=f(x)+f'(x)\Delta x+O(\Delta x)^2$$

$$f(x+\Delta x)-f(x)=f'(x)\Delta x+O(\Delta x)^2$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)+O(\Delta x)$$

Zápis pre deriváciu

Časť pre chybu

$$f(x-\Delta x)=f(x)-f'(x)\Delta x+\frac{f''(x)}{2!}(\Delta x)^2-\frac{f'''(x)}{3!}(\Delta x)^3+\frac{f''''(x)}{4!}(\Delta x)^4\ldots$$

... ekvivalentne

Presnosť výpočtu pokr.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \frac{f'''(x)}{3!}(\Delta x)^3 + \frac{f''''(x)}{4!}(\Delta x)^4 \ldots$$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 - \frac{f'''(x)}{3!}(\Delta x)^3 + \frac{f''''(x)}{4!}(\Delta x)^4 \ldots$$

Po odpočítaní oboh rovníc...

$$f(x+\Delta x) - f(x-\Delta x) = 2f'(x)\Delta x + O(\Delta x)^3$$

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + O(\Delta x)^2$$

Zápis pre deriváciu

Časť pre chybu
(mení sa kvadraticky)

VYHODNOTENIE DRUHEJ DERIVÁCIE

Druhá derivácia

Pomôžeme si opäť Taylorovým rozvojom

$$f(x+\Delta x)=f(x)+f'(x)\Delta x+\frac{f''(x)}{2!}(\Delta x)^2+\frac{f'''(x)}{3!}(\Delta x)^3+\frac{f''''(x)}{4!}(\Delta x)^4\dots$$

$$f(x-\Delta x)=f(x)-f'(x)\Delta x+\frac{f''(x)}{2!}(\Delta x)^2-\frac{f'''(x)}{3!}(\Delta x)^3+\frac{f''''(x)}{4!}(\Delta x)^4\dots$$

Po spočítaní:

$$f(x+\Delta x)+f(x-\Delta x)=2f(x)+\frac{2f''(x)}{2!}(\Delta x)^2+O(\Delta x)^4\dots$$

$$f''(x)=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}+\cancel{\frac{O(\Delta x)^4}{(\Delta x)^2}}\dots$$

$$f''(x)\cong\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}$$

Príklad na druhú deriváciu

$$f(x) = 2e^{1.5x}$$

$$f''(3)_{approx} = ???$$ ← $\Delta x = 0.1$

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

$$f''(x) \cong \frac{f(3+0.1) - 2f(3) + f(3-0.1)}{(0.1)^2}$$

$$= \frac{2e^{1.5 \cdot 3.1} - 2\left(2e^{1.5 \cdot 3}\right) + 2e^{1.5 \cdot 2.9}}{(0.1)^2} = 405.84$$

$$E_t = |405.08 - 405.84| = 0.76$$

$$E_r = 0.187\%$$

$$f(x) = 2e^{1.5x}$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$f'(x) = 2 \cdot 1.5 \cdot e^{1.5x}$$

$$f''(x) = 2 \cdot (1.5)^2 \cdot e^{1.5x}$$

$$= 4.5e^{1.5x}$$

$$f''(3)_{real} = 4.5e^{1.5 \cdot 3}$$

$$= 405.08$$

THE END

Stanislav.Antalic@fmph.uniba.sk